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Edge metric dimension of some classes of circulant graphs

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Abstract

Let $G = (V(G), E(G))$ be a connected graph and $x, y \in V(G)$, $d(x, y) = \min\{$ length of $x - y$ path $\}$ and for $e \in E(G)$, $d(x, e) = \min\{d(x, a), d(x, b)\}$, where $e = ab$. A vertex x distinguishes two edges e_1 and e_2, if $d(e_1, x) \neq d(e_2, x)$. Let $W_E = \{w_1, w_2, \ldots, w_k\}$ be an ordered set in $V(G)$ and let $e \in E(G)$. The representation $r(e \mid W_E)$ of e with respect to W_E is the k-tuple $(d(e, w_1), d(e, w_2), \ldots, d(e, w_k))$. If distinct edges of G have distinct representation with respect to W_E, then W_E is called an edge metric generator for G. An edge metric generator of minimum cardinality is an edge metric basis for G, and its cardinality is called edge metric dimension of G, denoted by $\mathrm{edim}(G)$. The circulant graph $C_n(1, m)$ has vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1\} \cup \{v_i v_{i+m} : 1 \leq i \leq n-m\} \cup \{v_{n-m+i} v_i : 1 \leq i \leq m\}$. In this paper, it is shown that the edge metric dimension of circulant graphs $C_n(1, 2)$ and $C_n(1, 3)$ is constant.

1 Introduction and Preliminaries

Suppose G is a connected and undirected graph with vertex set $V(G)$ and edge set $E(G)$. The degree d_v of a vertex $v \in V(G)$ is the total number of vertices joining to v. The maximum and minimum degree of the graph G are represented by $\Delta(G)$ and $\delta(G)$ respectively. Slater introduced the concept of metric

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dimension in order to locate an intruder's place in a network [18]. Melter and Harary further discussed the Slater's concept in [4]. The metric dimension, is defined as follows: Let $W = \{v_1, v_2, \dots, v_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The representation $r(v \mid W)$ of v with respect to W is the k-tuple $(d(v, v_1), d(v, v_2), \dots, d(v, v_k))$. If distinct vertices of G have distinct representation with respect to W, then W is called resolving set for G. A resolving set of minimum cardinality is a metric basis for G, and its cardinality is called the metric dimension of G, denoted by $\dim(G)$. For more detail on metric dimension, we refer to [5, 9, 20].
Metric dimension is an essential tool in image processing and pattern recognition (see [14]). In games such as mastermind and coin weighing, the use of metric dimension of graphs was also studied in [1]. In pharmaceutical chemistry, metric dimension of graphs provides distinct representation of different chemical compounds (see [2, 10]). The vertices and edges represent the atoms and bond types of different chemicals respectively. In this way, we can show the structure of chemical compounds by these labeled graphs. For applications in robot navigation, combinatorial optimization and coast guard loran, we refer articles [17, 18, 19].
Kelenc in [11] extended the idea of metric dimension to edge metric dimension. Let $d(x, e)$ denotes the distance between edge e and vertex x, defined as $d(x, e) = \min\{d(x, a), d(x, b)\}$, where $e = ab$ (see [11]). A vertex x distinguishes two edges e_1 and e_2, if $d(e_1, x) \neq d(e_2, x)$. Let $W_E = \{w_1, w_2, \dots, w_k\}$ be an ordered set in $V(G)$ and let $e \in E(G)$. The representation $r(e \mid W_E)$ of e with respect to W_E is the k-tuple $(d(e, w_1), d(e, w_2), \dots, d(e, w_k))$. If distinct edges of G have distinct representation with respect to W_E, then W_E is called an edge metric generator for G (see [11]). An edge metric generator of minimum cardinality is an edge metric basis for G, and its cardinality is called edge metric dimension of G, denoted by $\mathrm{edim}(G)$. Kelenc in [11] also computed the edge metric dimension for path, cycle, wheel and complete graphs. In [21], Zubrilina classified the graphs on n vertices for which edge metric dimension is $n-1$. In [13], Kratica computed the edge metric dimension of generalized Petersen graphs $GP(n, k)$ for $k = 1, 2$ while for the other values of k the lower bound is given. In 2018, Mufti et al. computed the edge metric dimension of barycentric subdivision of Cayley graphs (see [15]). The following lemmas are helpful for calculating the edge metric dimension of graphs:

Lemma 1.1. *[11] $edim(G) = 1$ if and only if G is the path graph.*

Lemma 1.2. *[11] For a connected graph G, $edim(G) \geq log_2(\triangle(G))$.*

Lemma 1.3. *[11] For a connected graph G, $edim(G) \geq 1 + \lceil \log_2 \delta(G) \rceil$.*

The circulant graph $C_n(1, m)$ has $V(C_n(1, m)) = \{v_1, v_2, \dots, v_n\}$ and $E(C_n(1, m)) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1\} \cup \{v_i v_{i+m} : 1 \leq i \leq$

$n-m\} \cup \{v_{n-m+i}v_i : 1 \leq i \leq m\}$. Circulant graphs are an important class of graphs used in computer sciences especially in designing of computer network topologies and local area networks.
The rest of paper is structured as follows:

In the second section, we will compute the edge metric dimension of the family of circulant graphs $C_n(1,2)$. In third section, we will calculate the edge metric dimension of the family of circulant graphs $C_n(1,3)$. In last section, we concluded the paper by open problem.

2 Edge metric dimension of family of circulant graphs $C_n(1,2)$

In this section, we will calculate the edge metric dimension of family of circulant graphs $C_n(1,2)$. Figure 1 shows the circulant graph $C_8(1,2)$.

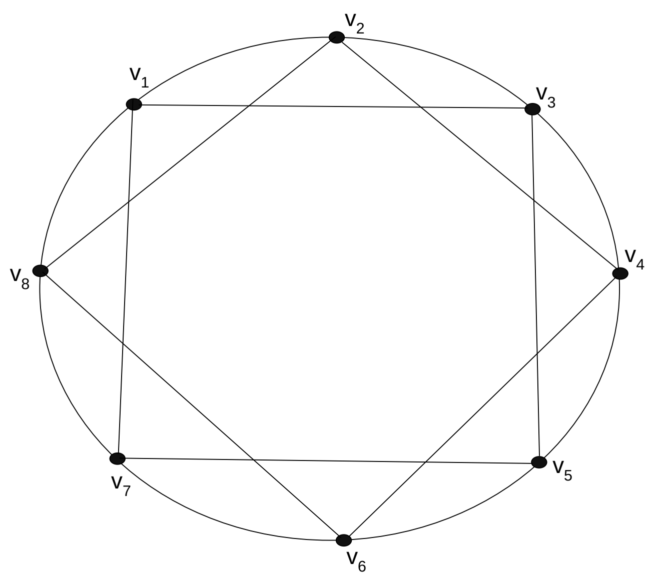

Figure 1: Circulant graph $C_8(1,2)$

The following theorem tells us the metric dimension of $C_n(1,2)$.

Theorem 2.1. *[7, 16] Let $C_n(1,2)$ be a circulant graph with $n \geq 5$, then*

$$dim(C_n(1,2)) = \begin{cases} 3, & \text{if } n \equiv 0,2,3 \ (mod\ 4); \\ \leq 4, & otherwise. \end{cases}$$

Now, we will compute the edge metric dimension of $C_n(1,2)$.

Theorem 2.2. *Let $C_n(1,2)$ be a circulant graph with $n \geq 5$, then*

$$edim(C_n(1,2)) = \begin{cases} 5, & \text{if } n \equiv 1,2 \ (mod\ 4); \\ 4, & otherwise. \end{cases}$$

Proof. In order to compute edge metric dimension of $C_n(1,2)$, we have the following cases.
Case (i): When $n \equiv 0 \pmod 4$ and $n \geq 5$.
Let $n = 4k$, $k \geq 2$, $k \in \mathbf{Z}^+$ and $W_E = \{v_1, v_2, v_{2k}, v_{2k+1}\} \subset V(C_n(1,2))$, we have to show that W_E is an edge metric generator of $C_n(1,2)$. For this, we give representations of each edge of $C_n(1,2)$.

$$r(v_{2i+1}v_{2i+2}|W_E) = \begin{cases} (i, i, k-i-1, k-i), & \text{if } 0 \leq i \leq k-1; \\ (k, k, 1, 0), & \text{if } i = k; \\ (2k-i, 2k-i, i-k+1, i-k), & \text{if } k+1 \leq i \leq 2k-1; \end{cases}$$

$$r(v_{2i+1}v_{2i+3}|W_E) = \begin{cases} (0, 1, k-1, k-1), & \text{if } i = 0; \\ (i, i, k-i-1, k-i-1), & \text{if } 1 \leq i \leq k-2; \\ (k-1, k-1, 1, 0), & \text{if } i = k-1; \\ (2k-i-1, 2k-i, i-k+1, i-k), & \text{if } k \leq i \leq 2k-2; \end{cases}$$

$$r(v_{n-1}v_1|W_E) = (0, 1, k, k-1),$$

$$r(v_{2i}v_{2i+1}|W_E) = \begin{cases} (i, i-1, k-i, k-i), & \text{if } 1 \leq i \leq k; \\ (2k-i, 2k-i+1, i-k, i-k), & \text{if } k+1 \leq i \leq 2k-1; \end{cases}$$

$$r(v_nv_1|W_E) = (0, 1, k, k),$$

$$r(v_{2i}v_{2i+2}|W_E) = \begin{cases} (i, i-1, k-i-1, k-i), & \text{if } 1 \leq i \leq k-1; \\ (k, k-1, 0, 1), & \text{if } i = k; \\ (2k-i, 2k-i, i-k, i-k), & \text{if } k+1 \leq i \leq 2k-1; \end{cases}$$

and $r(v_nv_2|W_E) = (1, 0, k-1, k)$.

We see that there are no two edges having the same representations. This shows that $\text{edim}(C_n(1,2)) \leq 4$.
On the other hand, we have to show that $\text{edim}(C_n(1,2)) \geq 4$. For this purpose, we have to show that there is no edge metric generator have cardinality 3, we suppose on contrary that $\text{edim}(C_n(1,2)) = 3$ and let $W_E = \{v_1, v_i, v_j\}$. Then the Table 1 shows all order pairs of edges (e, f) for which $r(e|W_E) = r(f|W_E)$.

Conditions on i and j	(e, f)
$2 \leq i \leq j \leq 2k$	$(v_nv_1, v_{n-1}v_1)$
$2 \leq i \leq 2k$, $2k+1 \leq j \leq n-1$ and j is even	$(v_nv_1, v_{n-1}v_1)$
$2 \leq i \leq 2k-1$ and $j = n$	$(v_{n-1}v_n, v_{n-2}v_n)$
$i = 2k$ and $j = n$	(v_1v_2, v_1v_3)
$2 \leq i \leq 2k-1$, $2k+1 \leq j \leq n-2$ and j is odd	$(v_{n-1}v_n, v_{n-2}v_n)$
$i = 2k, 2k+2 \leq j \leq n$ and j is odd	(v_1v_2, v_1v_3)
$i = 2k$ and $j = 2k+1$	$(v_{2k-1}v_{2k}, v_{2k-2}v_{2k})$
$2 \leq i \leq 2k-1$ and $j = n-1$	$(v_{2k}v_{2k+1}, v_{2k}v_{2k+2})$

Table 1: (e, f) for which $r(e|W_E) = r(f|W_E)$

In all possibilities, we conclude that there is no edge metric generator of 3 vertices. Hence $\text{edim}(C_n(1,2)) = 4$.
Case (ii): When $n \equiv 1 \pmod 4$ and $n \geq 5$.
Let $n = 4k+1$, $k \geq 1$, $k \in \mathbf{Z}^+$ and $W_E = \{v_1, v_2, v_3, v_{2k+2}, v_{2k+3}\} \subset V(C_n(1,2))$, we have to show that W_E is an edge metric generator of $C_n(1,2)$. For this, we give representations of each edge of $C_n(1,2)$.

$$r(v_{2i+1}v_{2i+2}|W_E) = \begin{cases} (0,0,1,k,k), & \text{if } i = 0; \\ (i,i,i-1,k-i,k-i+1), & \text{if } 1 \leq i \leq k; \\ (2k-i, 2k-i+1, 2k-i+1, \\ i-k, i-k-1), & \text{if } k+1 \leq i \leq 2k-1; \end{cases}$$

$r(v_nv_1|W_E) = (0, 1, 1, k, k-1)$,

$$r(v_{2i+1}v_{2i+3}|W_E) = \begin{cases} (0,1,0,k,k), & \text{if } i = 0; \\ (i,i,i-1,k-i,k-i), & \text{if } 1 \leq i \leq k-1; \\ (k,k,k-1,1,0), & \text{if } i = k; \\ (2k-i, 2k-i, 2k-i+1, \\ i-k, i-k-1), & \text{if } k+1 \leq i \leq 2k-1; \end{cases}$$

$r(v_nv_2|W_E) = (1, 0, 1, k, k-1)$,

$$r(v_{2i}v_{2i+1}|W_E) = \begin{cases} (i, i-1, i-1, k-i, k-i), & \text{if } 1 \leq i \leq k; \\ (k, k, k, 0, 0), & \text{if } i = k+1; \\ (2k-i+1, 2k-i+1, 2k-i+2, \\ i-k-1, i-k-1), & \text{if } k+2 \leq i \leq 2k; \end{cases}$$

$r(v_{n-1}v_1|W_E) = (0, 1, 1, k-1, k-1)$ and

$$r(v_{2i}v_{2i+2}|W_E) = \begin{cases} (1, 0, 1, k-1, k), & \text{if } i = 1; \\ (i, i-1, i-1, k-i, k-i+1), & \text{if } 2 \leq i \leq k; \\ (k-1, k, k, 0, 1), & \text{if } i = k+1; \\ (2k-i, 2k-i+1, 2k-i+1, \\ i-k-1, i-k-1), & \text{if } k+2 \leq i \leq 2k-1. \end{cases}$$

We see that there are no two edges having the same representations. This shows that $\text{edim}(C_n(1,2)) \leq 5$.
On the other hand, we have to show that $\text{edim}(C_n(1,2)) \geq 5$. For this purpose, we have to show that there is no edge metric generator have cardinality 4, we suppose on contrary that $\text{edim}(C_n(1,2)) = 4$ and let $W_E = \{v_1, v_i, v_j, v_l\}$. Then the Table 2 shows all order pairs of edges (e, f) for which $r(e|W_E) = r(f|W_E)$.

Conditions on i, j and l	(e, f)
$2 \leq i \leq j \leq l \leq 2k$	$(v_nv_1, v_{n-1}v_1)$
$2 \leq i \leq j \leq 2k, 2k+1 \leq l \leq n-2$ and l is even	$(v_{n-1}v_n, v_{n-2}v_n)$
$2 \leq i \leq j \leq 2k$ and $l = n-1$	$(v_{2k+1}v_{2k+2}, v_{2k+1}v_{2k+3})$
$2 \leq i \leq j \leq 2k, 2k+1 \leq l \leq n-1$ and l is odd	$(v_nv_1, v_{n-1}v_1)$
$2 \leq i \leq j \leq 2k$ and $l = n$	(v_nv_{n-1}, v_nv_{n-2})

Table 2: (e, f) for which $r(e|W_E) = r(f|W_E)$

Since in each case we get contradiction. We conclude that there is no edge metric generator of 4 vertices. So it is obvious that edge metric generator of 3 vertices does not exist. Hence $\text{edim}(C_n(1,2)) = 5$.
Case (iii): When $n \equiv 2 \pmod 4$ and $n \geq 5$.
Let $n = 4k+2$, $k \geq 2$, $k \in \mathbf{Z}^+$ and $W_E = \{v_1, v_2, v_3, v_{2k+2}, v_{2k+3}\} \subset V(C_n(1,2))$, we have to show that W_E is an edge metric generator of $C_n(1,2)$. For this, we give representations of each edge of $C_n(1,2)$.

$$r(v_{2i+1}v_{2i+2}|W_E) = \begin{cases} (0,0,1,k,k), & \text{if } i=0; \\ (i,i,i-1,k-i,k+1-i), & \text{if } 1 \le i \le k; \\ (k,k,k,1,2), & \text{if } i=k+1; \\ (2k-i,2k-i,2k-i+1, \\ i-k,i-k+1), & \text{if } k+2 \le i \le 2k; \end{cases}$$

$$r(v_{2i+1}v_{2i+3}|W_E) = \begin{cases} (0,1,0,k,k), & \text{if } i=0; \\ (i,i,i-1,k-i,k-i), & \text{if } 1 \le i \le k-1; \\ (k,k,k-1,1,0), & \text{if } i=k; \\ (2k-i,2k-i+1,2k-i+1, \\ i-k,i-k-1), & \text{if } k+1 \le i \le 2k-1; \end{cases}$$

$r(v_{n-1}v_1|W_E) = (0,1,1,k,k-1)$,

$$r(v_{2i}v_{2i+1}|W_E) = \begin{cases} (i,i-1,i-1,k+1-i,k+1-i), & \text{if } 1 \le i \le k; \\ (k,k,k,0,0), & \text{if } i=k+1; \\ (2k-i+1,2k-i+2,2k-i+2, \\ i-k-1,i-k-1), & \text{if } k+2 \le i \le 2k; \end{cases}$$

$r(v_nv_1|W_E) = (0,1,1,k,k)$,

$$r(v_{2i}v_{2i+2}|W_E) = \begin{cases} (1,0,1,k-1,k), & \text{if } i=1; \\ (i,i-1,i-1,k-i,k+1-i), & \text{if } 2 \le i \le k; \\ (k,k,k,0,1), & \text{if } i=k+1; \\ (2k+1-i,2k+1-i,2k-i+2, \\ i-k-1,i-k-1), & \text{if } k+2 \le i \le 2k; \end{cases}$$

and $\quad r(v_nv_2|W_E) = (1,0,1,k,k)$.

We see that there are no two edges having the same representations. This shows that $\mathrm{edim}(C_n(1,2)) \le 5$.
On the other hand, we have to show that $\mathrm{edim}(C_n(1,2)) \ge 5$. For this purpose, we have to show that there is no edge metric generator have cardinality 4, we suppose on contrary that $\mathrm{edim}(C_n(1,2)) = 4$ and let $W_E = \{v_1, v_i, v_j, v_l\}$. Then the Table 3 shows all order pairs of edges (e,f) for which $r(e|W_E) = r(f|W_E)$.

Conditions on i, j and l	(e, f)
$2 \leq i \leq j \leq l \leq 2k+1$	$(v_n v_1, v_{n-1} v_1)$
$2 \leq i \leq j \leq 2k+1, 2k+2 \leq l \leq n-1$ and l is even	$(v_n v_1, v_{n-1} v_1)$
$2 \leq i \leq j \leq 2k+1$ and $l = n$	$(v_{2k+1} v_{2k+2}, v_{2k+1} v_{2k+3})$
$2 \leq i \leq j \leq 2k+1, 2k+2 \leq l \leq n-2$ and l is odd	$(v_n v_{n-1}, v_n v_{n-2})$
$2 \leq i \leq j \leq 2k+1$ and $l = n-1$	$(v_{2k+2} v_{2k+3}, v_{2k+2} v_{2k+4})$

Table 3: (e, f) for which $r(e|W_E) = r(f|W_E)$

Since in each case we get contradiction. We conclude that there is no edge metric generator of 4 vertices. So it is obvious that edge metric generator of 3 vertices does not exist. Hence $\mathrm{edim}(C_n(1,2)) = 5$.

Case (iv): When $n \equiv 3 \pmod 4$ and $n \geq 5$.

Let $n = 4k+3$, $k \geq 2$, $k \in \mathbf{Z}^+$ and $W_E = \{v_1, v_2, v_{2k+1}, v_{2k+2}\} \subset V(C_n(1,2))$, we have to show that W_E is an edge metric generator of $C_n(1,2)$. For this, we give representations of each edge of $C_n(1,2)$.

$$r(v_{2i+1}v_{2i+2}|W_E) = \begin{cases} (i, i, k-i, k-i), & \text{if } 0 \leq i \leq k; \\ (2k+1-i, 2k-i+2, i-k, i-k), & \text{if } k+1 \leq i \leq 2k; \end{cases}$$

$r(v_n v_1|W_E) = (0, 1, k, k+1)$,

$$r(v_{2i+1}v_{2i+3}|W_E) = \begin{cases} (0, 1, k-1, k), & \text{if } i = 0; \\ (i, i, k-i-1, k-i), & \text{if } 1 \leq i \leq k-1; \\ (k, k, 0, 1), & \text{if } i = k; \\ (2k+1-i, 2k+1-i, i-k, i-k), & \text{if } k+1 \leq i \leq 2k; \end{cases}$$

$r(v_n v_2|W_E) = (1, 0, k, k)$,

$$r(v_{2i}v_{2i+1}|W_E) = \begin{cases} (i, i-1, k-i, k+1-i), & \text{if } 1 \leq i \leq k; \\ (k+1, k, 1, 0), & \text{if } i = k+1; \\ (2k-i+2, 2k-i+2, i-k, \\ i-k-1), & \text{if } k+2 \leq i \leq 2k+1; \end{cases}$$

$r(v_{n-1}v_1|W_E) = (0, 1, k, k)$ and

$$r(v_{2i}v_{2i+2}|W_E) = \begin{cases} (i, i-1, k-i, k-i), & \text{if } 1 \leq i \leq k-1; \\ (k, k-1, 1, 0), & \text{if } i = k; \\ (k, k, 1, 0), & \text{if } i = k+1; \\ (2k+1-i, 2k-i+2, i-k, \\ i-k-1), & \text{if } k+2 \leq i \leq 2k. \end{cases}$$

We see that there are no two edges having the same representations. This shows that $\mathrm{edim}(C_n(1,2)) \leq 4$.
On the other hand, we have to show that $\mathrm{edim}(C_n(1,2)) \geq 4$. For this purpose, we have to show that there is no edge metric generator have cardinality 3, we suppose on contrary that $\mathrm{edim}(C_n(1,2)) = 3$ and let $W_E = \{v_1, v_i, v_j\}$. Then the Table 4 shows all order pairs of edges (e, f) for which $r(e|W_E) = r(f|W_E)$.

Conditions on i and j	(e, f)
$2 \leq i \leq j \leq 2k+1$	$(v_nv_1, v_{n-1}v_1)$
$2 \leq i \leq 2k, 2k+2 \leq j \leq n-2$ and j is even	(v_nv_{n-1}, v_nv_{n-2})
$i = 2k+1, 2k+3 \leq j \leq n$ and j is even	$(v_{2k+1}v_{2k+2}, v_{2k+1}v_{2k+3})$
$i = 2k+1$ and $j = 2k+2$	(v_nv_2, v_nv_{n-2})
$2 \leq i \leq 2k+1$ and $j = n-1$	$(v_{2k+1}v_{2k+2}, v_{2k+1}v_{2k+3})$
$2 \leq i \leq 2k+1, 2k+2 \leq j \leq n-1$ and j is odd	$(v_nv_1, v_{n-1}v_1)$
$2 \leq i \leq 2k+1, j = n$ and $i \neq 2k$	$(v_{2k}v_{2k+2}, v_{2k+2}v_{2k+4})$
$i = 2k$ and $j = n$	$(v_{2k}v_{2k+1}, v_{2k}v_{2k+2})$

Table 4: (e, f) for which $r(e|W_E) = r(f|W_E)$

In all possibilities, we conclude that there is no edge metric generator of 3 vertices. Hence $\mathrm{edim}(C_n(1,2)) = 4$.

□

3 Edge metric dimension of family of circulant graphs $C_n(1,3)$

In this section, we will compute the edge metric dimension of family of circulant graphs $C_n(1,3)$. Figure 2 shows the circulant graph $C_{16}(1,3)$.

The following theorem tells us the metric dimension of $C_n(1,3)$.

Theorem 3.1. *[6, 8, 16] Let $C_n(1,3)$ be a circulant graph with $n \geq 6$, then*

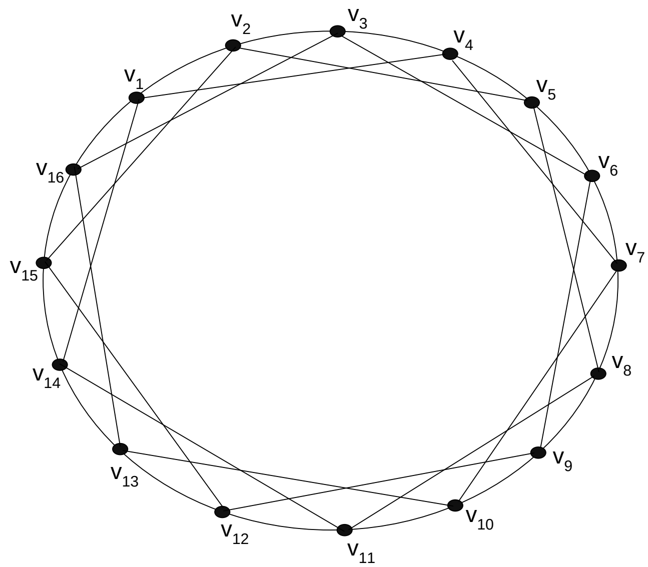

Figure 2: Circulant graph $C_{16}(1,3)$

$$dim(C_n(1,3)) = \begin{cases} 3, & \text{if } n \equiv 1 \ (mod\ 6); \\ 4, & \text{if } n \equiv 0,3,4,5 \ (mod\ 6); \\ 4 \leq dim(C_n(1,3)) \leq 6, & \text{if } n \equiv 2 \ (mod\ 6); \end{cases}$$

Now, we will compute the edge metric dimension of $C_n(1,3)$.

Theorem 3.2. *Let $C_n(1,3)$ be a circulant graph with $n \geq 6$, then*

$$edim(C_n(1,3)) = \begin{cases} 6, & n = 8 \\ 5, & \text{if } n \equiv 2,5 \ (mod\ 6) \ and\ n \neq 8; \\ 4, & otherwise; \end{cases}$$

Proof. In order to compute edge metric dimension of $C_n(1,3)$, we have the following cases.

Case (i): When $n=7$, $W_E=\{v_1,v_2,v_4,v_5\} \subset V(C_7(1,3))$ is edge metric generator and hence $\mathrm{edim}(C_7(1,3))=4$.

Case (ii): When $n=8$, $W_E=\{v_1,v_2,v_3,v_4,v_5,v_6\} \subset V(C_8(1,3))$ is edge metric generator and hence $\mathrm{edim}(C_8(1,3))=6$.

Case (iii): When $n \equiv 0 \pmod 6$ and $n \geq 6$.
Let $n=6k$, $k\geq 1$, $k \in \mathbf{Z}^+$ and $W_E=\{v_1,v_2,v_3,v_4\} \subset V(C_n(1,3))$, we have to show that W_E is an edge metric generator of $C_n(1,3)$. For this, we give representations of each edge of $C_n(1,3)$.

$$r(v_{3i+1}v_{3i+2}|W_E)=\begin{cases}(0,0,1,1), & \text{if } i=0;\\ (i,i,i,i-1), & \text{if } 1\leq i\leq k;\\ (2k-i,2k-i,2k-i+1,\\ 2k-i+1), & \text{if } k+1\leq i\leq 2k-1;\end{cases}$$

$$r(v_{3i+2}v_{3i+3}|W_E)=\begin{cases}(1,0,0,1), & \text{if } i=0;\\ (i+1,i,i,i), & \text{if } 1\leq i\leq k-1;\\ (k,k,k,k), & \text{if } i=k;\\ (2k-i,2k-i,2k-i,2k-i+1), & \text{if } k+1\leq i\leq 2k;\end{cases}$$

$$r(v_{3i}v_{3i+1}|W_E)=\begin{cases}(i,i,i-1,i-1), & \text{if } 1\leq i\leq k;\\ (2k-i,2k-i+1,2k-i+1,\\ 2k-i+1), & \text{if } k+1\leq i\leq 2k-1;\end{cases}$$

$r(v_nv_1|W_E)=(0,1,1,1)$,

$$r(v_{3i+1}v_{3i+4}|W_E)=\begin{cases}(0,1,1,0), & \text{if } i=0;\\ (i,i+1,i,i-1), & \text{if } 1\leq i\leq k-1;\\ (k-1,k,k,k-1), & \text{if } i=k;\\ (2k-i-1,2k-i,2k-i+1,\\ 2k-i), & \text{if } k+1\leq i\leq 2k-1;\end{cases}$$

$r(v_{n-2}v_1|W_E)=(0,1,2,1)$,

$$r(v_{3i+2}v_{3i+5}|W_E) = \begin{cases} (1,0,1,1), & \text{if } i=0; \\ (i+1,i,i+1,i), & \text{if } 1 \le i \le k-1; \\ (k,k-1,k,k), & \text{if } i=k; \\ (2k-i,2k-i-1,2k-i, \\ 2k-i+1), & \text{if } k+1 \le i \le 2k-2; \end{cases}$$

$r(v_{n-1}v_2|W_E) = (1,2,1,2)$,

$$r(v_{3i}v_{3i+3}|W_E) = \begin{cases} (i+1,i,i-1,i), & \text{if } 1 \le i \le k-1; \\ (k,k,k-1,k), & \text{if } i=k; \\ (2k-i,2k-i+1,2k-i, \\ 2k-i+1), & \text{if } k+1 \le i \le 2k-1; \end{cases}$$

and $r(v_nv_3|W_E) = (1,1,0,1)$.

We see that there are no two edges having the same representations. This shows that $\text{edim}(C_n(1,3)) \le 4$.

On the other hand, we have to show that $\text{edim}(C_n(1,3)) \ge 4$. For this purpose, we have to show that there is no edge metric generator have cardinality 3, we suppose on contrary that $\text{edim}(C_n(1,3)) = 3$ and let $W_E = \{v_1, v_i, v_j\}$. Then the Table 5 shows all order pairs of edges (e,f) for which $r(e|W_E) = r(f|W_E)$.

Conditions on i and j	(e,f)
$2 \le i \le j \le \frac{n}{2}$	$(v_nv_1, v_{n-2}v_1)$
$2 \le i \le \frac{n}{2}, \frac{n}{2}+1 \le j \le n$ and $j \ne \{\frac{n}{2}+1, \frac{n}{2}+4, \frac{n}{2}+7, \ldots, n-2, n\}$	$(v_nv_1, v_{n-2}v_1)$
$2 \le i \le \frac{n}{2}-1, \frac{n}{2}+1 \le j \le n$ and $j \ne \{\frac{n}{2}+3, \frac{n}{2}+6, \frac{n}{2}+9, \ldots, n-3, n-1\}$	$(v_{n-1}v_n, v_{n-3}v_n)$
$i = \frac{n}{2}, \frac{n}{2}+1 \le j \le n$ and $j \ne \{\frac{n}{2}+3, \frac{n}{2}+6, \frac{n}{2}+9, \ldots, n-3, n-1\}$	(v_1v_2, v_1v_4)

Table 5: (e,f) for which $r(e|W_E) = r(f|W_E)$

In all possibilities, we get contradiction so it concludes that there is no edge metric generator of 3 vertices. Hence $\text{edim}(C_n(1,3)) = 4$, for $n \equiv 0 \pmod 6$.

Case (iv): When $n \equiv 1 \pmod 6$ and $n \ge 13$.

Let $n = 6k+1$, $k \ge 2$, $k \in \mathbf{Z}^+$ and $W_E = \{v_1, v_3, v_5, v_{n-1}\} \subset V(C_n(1,3))$, we have to show that W_E is an edge metric generator of $C_n(1,3)$. For this, following are the representations of each edge of $C_n(1,3)$.

$$r(v_{3i+1}v_{3i+2}|W_E) = \begin{cases} (0,1,1,1), & \text{if } i=0; \\ (i,i,i-1,i+1), & \text{if } 1 \le i \le k-1; \\ (k,k,k-1,k), & \text{if } i=k; \\ (k-1,k,k,k-1), & \text{if } i=k+1; \\ (2k-i,2k-i+1,2k-i+2, \\ 2k-i), & \text{if } k+2 \le i \le 2k-1; \end{cases}$$

$r(v_n v_1|W_E) = (0,1,2,1)$,

$$r(v_{3i+2}v_{3i+3}|W_E) = \begin{cases} (1,0,1,1), & \text{if } i=0; \\ (i+1,i,i-1,i+1), & \text{if } 1 \le i \le k-1; \\ (k,k,k-1,k-1), & \text{if } i=k; \\ (2k-i,2k-i+1,2k-i+1, \\ 2k-i-1), & \text{if } k+1 \le i \le 2k-1; \end{cases}$$

$$r(v_{3i}v_{3i+1}|W_E) = \begin{cases} (1,0,1,2), & \text{if } i=1; \\ (i,i-1,i-1,i+1), & \text{if } 2 \le i \le k-1; \\ (k,k-1,k-1,k), & \text{if } i=k; \\ (k,k,k,k-1), & \text{if } i=k+1; \\ (2k-i+1,2k-i+1,2k-i+2, \\ 2k-i), & \text{if } k+2 \le i \le 2k; \end{cases}$$

$$r(v_{3i+1}v_{3i+4}|W_E) = \begin{cases} (0,1,1,2), & \text{if } i=0; \\ (i,i,i,i+2), & \text{if } 1 \le i \le k-1; \\ (k,k,k,k), & \text{if } i=k; \\ (2k-i,2k-i,2k-i+2, \\ 2k-i), & \text{if } k+1 \le i \le 2k-1; \end{cases}$$

$r(v_n v_3|W_E) = (1,0,2,1)$,

$$r(v_{3i+2}v_{3i+5}|W_E) = \begin{cases} (1,1,0,1), & \text{if } i=0; \\ (i+1,i+1,i-1,i+1), & \text{if } 1 \le i \le k-1; \\ (2k-i-1,2k-i+1,2k-i-1, \\ 2k-i-1), & \text{if } k \le i \le 2k-3; \\ (1,3,3,1), & \text{if } i=2k-2; \end{cases}$$

$r(v_{n-2}v_1|W_E) = (0,2,2,1)$,

$$r(v_{3i}v_{3i+3}|W_E) = \begin{cases} (2,0,1,2), & \text{if } i=1; \\ (i+1,i-1,i-1,i+1), & \text{if } 2 \le i \le k-1; \\ (k+1,k-1,k-1,k-1), & \text{if } i=k; \\ (2k-i+1,2k-i+1,2k-i+1, & \\ 2k-i-1), & \text{if } k+1 \le i \le 2k-1; \end{cases}$$

and $r(v_{n-1}v_2|W_E) = (1,1,1,0)$.

We see that there are no two edges having the same representations. This shows that $\text{edim}(C_n(1,3)) \le 4$.

On the other hand, we have to show that $\text{edim}(C_n(1,3)) \ge 4$. For this purpose, we have to show that there is no edge metric generator have cardinality 3, we suppose on contrary that $\text{edim}(C_n(1,3)) = 3$ and let $W_E = \{v_1, v_i, v_j\}$. Then the Table 6 shows all order pairs of edges (e,f) for which $r(e|W_E) = r(f|W_E)$.

Conditions on i,j	(e,f)
$2 \le i \le j \le \frac{n+1}{2}$	$(v_nv_1, v_{n-2}v_1)$
$2 \le i \le \frac{n-1}{2}$, $\frac{n+1}{2} \le j \le n$ and $j \ne \{\frac{n+1}{2}+1, \frac{n+1}{2}+4, \frac{n+1}{2}+7, \ldots, n-2, n\}$	$(v_nv_1, v_{n-2}v_1)$
$2 \le i \le \frac{n-1}{2}$, $\frac{n+1}{2} \le j \le n$ and $j \ne \{\frac{n+1}{2}+3, \frac{n+1}{2}+6, \frac{n+1}{2}+9, \ldots, n-3, n-1\}$	$(v_{n-1}v_n, v_{n-3}v_n)$

Table 6: (e,f) for which $r(e|W_E) = r(f|W_E)$

In all possibilities, we get contradiction so it concludes that there is no edge metric generator of 3 vertices. Hence $\text{edim}(C_n(1,3)) = 4$, for $n \equiv 1 \pmod 6$.

Case (v): When $n \equiv 2 \pmod 6$ and $n \ge 14$.

Let $n = 6k+2$, $k \ge 2$, $k \in \mathbf{Z}^+$ and $W_E = \{v_1, v_2, v_{\frac{n}{2}-2}, v_{\frac{n}{2}-1}, v_{n-2}\} \subset V(C_n(1,3))$, we have to show that W_E is an edge metric generator of $C_n(1,3)$. For this, we give representations of each edge of $C_n(1,3)$.

$$r(v_{3i+1}v_{3i+2}|W_E) = \begin{cases} (i,i,k-i-1,k-i,i+1), & \text{if } 0 \le i \le k-1; \\ (k,k,1,1,k), & \text{if } i=k; \\ (2k-i+1,2k-i+1,i-k+1, & \\ i-k+1,2k-i), & \text{if } k+1 \le i \le 2k-1; \\ (1,1,k,k,1), & \text{if } i=2k; \end{cases}$$

$$r(v_{3i+2}v_{3i+3}|W_E) = \begin{cases} (i+1, i, k-i-1, k-i-1, i+2), & \text{if } 0 \le i \le k-2; \\ (k, k-1, 0, 0, k), & \text{if } i = k-1; \\ (k, k, k+1, 1, k-1), & \text{if } i = k; \\ (2k-i, 2k-i+1, i-k+1, \\ i-k+1, 2k-i-1), & \text{if } k+1 \le i \le 2k-1; \end{cases}$$

$r(v_n v_1|W_E) = (0, 1, k, k, 1)$,

$$r(v_{3i}v_{3i+1}|W_E) = \begin{cases} (i, i, k-i, k-i, i+1), & \text{if } 1 \le i \le k-1; \\ (k, k, 1, 0, k), & \text{if } i = k; \\ (2k-i+1, 2k-i+1, i-k+1, \\ i-k, 2k-i), & \text{if } k+1 \le i \le 2k-1; \\ (1, 1, k, k, 0), & \text{if } i = 2k; \end{cases}$$

$$r(v_{3i+1}v_{3i+4}|W_E) = \begin{cases} (i, i+1, k-i-1, k-i, i+1), & \text{if } 0 \le i \le k-2; \\ (k-1, k, 1, 1, k), & \text{if } i = k-1; \\ (k, k, 2, 1, k), & \text{if } i = k; \\ (2k-i+1, 2k-i, i-k+2, \\ i-k+1, 2k-i), & \text{if } k+1 \le i \le 2k-2; \\ (2, 1, k, k, 1), & \text{if } i = 2k-1; \end{cases}$$

$r(v_{n-1}v_2|W_E) = (1, 0, k-1, k, 1)$,

$$r(v_{3i+2}v_{3i+5}|W_E) = \begin{cases} (i+1, i, k-i-2, k-i-1, i+2), & \text{if } 0 \le i \le k-2; \\ (k, k-1, 0, 1, k), & \text{if } i = k-1; \\ (k, k, 1, 2, k-1), & \text{if } i = k; \\ (2k-i, 2k-i+1, i-k+1, \\ i-k+2, 2k-i-1), & \text{if } k+1 \le i \le 2k-2; \\ (1, 2, k, k, 1), & \text{if } i = 2k-1; \end{cases}$$

$r(v_n v_3|W_E) = (1, 1, k, k-1, 2)$,

$$r(v_{3i}v_{3i+3}|W_E) = \begin{cases} (i+1, i, k-i, k-i-1, i+2), & \text{if } 1 \le i \le k-2; \\ (k, k-1, 1, 0, k), & \text{if } i = k-1; \\ (k, k, 1, 0, k-1), & \text{if } i = k; \\ (2k-i, 2k-i+1, i-k+1, i-k, \\ 2k-i-1), & \text{if } k+1 \le i \le 2k-1; \end{cases}$$

and $\quad r(v_{n-2}v_1|W_E) = (0, 1, k, k, 0)$.

We see that there are no two edges having the same representations. This shows that $\text{edim}(C_n(1,3)) \leq 5$.
On the other hand, we have to show that $\text{edim}(C_n(1,3)) \geq 5$. For this purpose, we have to show that there is no edge metric generator have cardinality 4, we suppose on contrary that $\text{edim}(C_n(1,3)) = 4$ and let $W_E = \{v_1, v_i, v_j, v_l\}$. Then the Table 7 shows all order pairs of edges (e, f) for which $r(e|W_E) = r(f|W_E)$.

Conditions on i, j and l	(e, f)
$2 \leq i \leq j \leq l \leq \frac{n}{2}$ and $j, l \neq \frac{n}{2} - 1$	$(v_n v_1, v_{n-2} v_1)$
$2 \leq i \leq j \leq l \leq \frac{n}{2}$ and $j, l \neq \frac{n}{2} - 2$	$(v_{n-1} v_n, v_{n-3} v_n)$
$2 \leq i \leq j \leq l \leq \frac{n}{2}$ and $i \neq 2$	$(v_{\frac{n}{2}} v_{\frac{n}{2}+1}, v_{\frac{n}{2}} v_{\frac{n}{2}+3})$
$2 \leq i \leq j \leq l \leq \frac{n}{2}$ and $i \neq 3$	$(v_{\frac{n}{2}+1} v_{\frac{n}{2}+2}, v_{\frac{n}{2}+1} v_{\frac{n}{2}+4})$
$2 \leq i \leq j \leq \frac{n}{2}, \frac{n}{2} + 1 \leq l \leq n$ and $j \neq \frac{n}{2} - 1,$ also $l \neq \{\frac{n}{2} + 2, \frac{n}{2} + 5, \frac{n}{2} + 8, \ldots, n - 2, n\}$	$(v_n v_1, v_{n-2} v_1)$
$2 \leq i \leq j \leq \frac{n}{2}, \frac{n}{2} + 1 \leq l \leq n$ and $j \neq \frac{n}{2} - 2,$ also $l \neq \{\frac{n}{2} + 1, \frac{n}{2} + 4, \frac{n}{2} + 7, \ldots, n - 3, n - 1\}$	$(v_{n-1} v_n, v_{n-3} v_n)$
$2 \leq i \leq j \leq \frac{n}{2}, \frac{n}{2} + 1 \leq l \leq n$ and $i \neq 2,$ also $l \neq \{\frac{n}{2} + 1, \frac{n}{2} + 3, \frac{n}{2} + 6, \frac{n}{2} + 9, \ldots, n - 1\}$	$(v_{\frac{n}{2}} v_{\frac{n}{2}+1}, v_{\frac{n}{2}} v_{\frac{n}{2}+3})$
$2 \leq i \leq j \leq \frac{n}{2}, \frac{n}{2} + 1 \leq l \leq n$ and $i \neq 3,$ also $l \neq \{\frac{n}{2} + 2, \frac{n}{2} + 4, \frac{n}{2} + 7, \frac{n}{2} + 10, \ldots, n\}$	$(v_{\frac{n}{2}+1} v_{\frac{n}{2}+2}, v_{\frac{n}{2}+1} v_{\frac{n}{2}+4})$

Table 7: (e, f) for which $r(e|W_E) = r(f|W_E)$

In all possibilities, we get contradiction so it concludes that there is no edge metric generator of 4 vertices. Hence $\text{edim}(C_n(1,3)) = 5$, for $n \equiv 2 \pmod 6$.
Case (vi): When $n \equiv 3 \pmod 6$ and $n \geq 9$.
Let $n = 6k + 3$, $k \geq 1$, $k \in \mathbf{Z}^+$ and $W_E = \{v_1, v_2, v_3, v_4\} \subset V(C_n(1,3))$, we have to show that W_E is an edge metric generator of $C_n(1,3)$. For this, following are the representations of each edge of $C_n(1,3)$.

$$r(v_{3i+1} v_{3i+2}|W_E) = \begin{cases} (0,0,1,1), & \text{if } i = 0; \\ (i,i,i,i-1), & \text{if } 1 \leq i \leq k; \\ (k,k,k+1,k), & \text{if } i = k+1; \\ (2k-i+1, 2k-i+1, 2k-i+2, \\ 2k-i+2), & \text{if } k+2 \leq i \leq 2k; \end{cases}$$

$$r(v_{3i+2}v_{3i+3}|W_E) = \begin{cases} (1,0,0,1), & \text{if } i=0; \\ (i+1,i,i,i), & \text{if } 1 \le i \le k; \\ (2k-i+1,2k-i+1,2k-i+1, \\ 2k-i+2), & \text{if } k+1 \le i \le 2k; \end{cases}$$

$$r(v_{3i}v_{3i+1}|W_E) = \begin{cases} (i,i,i-1,i-1), & \text{if } 1 \le i \le k; \\ (k,k+1,k,k), & \text{if } i=k+1; \\ (2k-i+1,2k-i+2,2k-i+2, \\ 2k-i+2), & \text{if } k+2 \le i \le 2k; \end{cases}$$

$r(v_nv_1|W_E) = (0,1,1,1)$,

$$r(v_{3i+1}v_{3i+4}|W_E) = \begin{cases} (0,1,1,0), & \text{if } i=0; \\ (i,i+1,i,i-1), & \text{if } 1 \le i \le k; \\ (2k-i,2k-i+1,2k-i+2, \\ 2k-i+1), & \text{if } k+1 \le i \le 2k-1; \end{cases}$$

$r(v_{n-2}v_1|W_E) = (0,1,2,1)$,

$$r(v_{3i+2}v_{3i+5}|W_E) = \begin{cases} (1,0,1,1), & \text{if } i=0; \\ (i+1,i,i+1,i), & \text{if } 1 \le i \le k; \\ (2k-i+1,2k-i,2k-i+1, \\ 2k-i+2), & \text{if } k+1 \le i \le 2k; \end{cases}$$

$r(v_{n-1}v_2|W_E) = (1,0,1,2)$,

$$r(v_{3i}v_{3i+3}|W_E) = \begin{cases} (i+1,i,i-1,i), & \text{if } 1 \le i \le k; \\ (2k-i+1,2k-i+2,2k-i+1, \\ 2k-i+2), & \text{if } k+1 \le i \le 2k; \end{cases}$$

and $r(v_nv_3|W_E) = (1,1,0,1)$.

We see that there are no two edges having the same representations. This shows that $\mathrm{edim}(C_n(1,3)) \le 4$.

On the other hand, we have to show that $\mathrm{edim}(C_n(1,3)) \ge 4$. For this purpose, we have to show that there is no edge metric generator have cardinality 3, we suppose on contrary that $\mathrm{edim}(C_n(1,3)) = 3$ and let $W_E = \{v_1, v_i, v_j\}$. Then the Table 8 shows all order pairs of edges (e,f) for which $r(e|W_E) = r(f|W_E)$.

Conditions on i and j	(e,f)
$2 \le i \le j \le \frac{n+1}{2}$	$(v_n v_1, v_{n-2} v_1)$
$2 \le i \le \frac{n-1}{2}$, $\frac{n+1}{2} \le j \le n$ and $j \ne \{\frac{n+1}{2}+2, \frac{n+1}{2}+5, \frac{n+1}{2}+8, \ldots, n-2, n\}$	$(v_n v_1, v_{n-2} v_1)$
$2 \le i \le \frac{n-1}{2}$, $\frac{n+1}{2} \le j \le n$ and $j \ne \{\frac{n+1}{2}+1, \frac{n+1}{2}+4, \frac{n+1}{2}+7, \ldots, n-3, n-1\}$	$(v_{n-1} v_n, v_{n-3} v_n)$

Table 8: (e,f) for which $r(e|W_E) = r(f|W_E)$

In all possibilities, we get contradiction so it concludes that there is no edge metric generator of 3 vertices. Hence $\mathrm{edim}(C_n(1,3)) = 4$, for $n \equiv 3 \pmod 6$.

Case (vii): When $n \equiv 4 \pmod 6$ and $n \ge 10$.

Let $n = 6k+4$, $k \ge 1$, $k \in \mathbf{Z}^+$ and $W_E = \{v_1, v_2, v_3, v_4\} \subset V(C_n(1,3))$, we have to show that W_E is an edge metric generator of $C_n(1,3)$. For this, following are the representations of each edge of $C_n(1,3)$.

$$r(v_{3i+1}v_{3i+2}|W_E) = \begin{cases} (0,0,1,1), & \text{if } i=0; \\ (i,i,i,i-1), & \text{if } 1 \le i \le k; \\ (k,k+1,k+1,k), & \text{if } i=k+1; \\ (2k-i+1, 2k-i+2, 2k-i+2, \\ 2k-i+2), & \text{if } k+2 \le i \le 2k; \end{cases}$$

$r(v_n v_1|W_E) = (0,1,1,1)$,

$$r(v_{3i+2}v_{3i+3}|W_E) = \begin{cases} (1,0,0,1), & \text{if } i=0; \\ (i+1,i,i,i), & \text{if } 1 \le i \le k; \\ (2k-i+1, 2k-i+1, 2k-i+2, \\ 2k-i+2), & \text{if } k+1 \le i \le 2k; \end{cases}$$

$$r(v_{3i}v_{3i+1}|W_E) = \begin{cases} (i,i,i-1,i-1), & \text{if } 1 \le i \le k; \\ (2k-i+2, 2k-i+2, 2k-i+2, \\ 2k-i+3), & \text{if } k+2 \le i \le 2k; \end{cases}$$

$$r(v_{3i+1}v_{3i+4}|W_E) = \begin{cases} (0,1,1,0), & \text{if } i=0; \\ (i,i+1,i,i-1), & \text{if } 1 \le i \le k; \\ (k,k+1,k,k), & \text{if } i=k+1; \\ (2k-i+1, 2k-i+2, 2k-i+1, \\ 2k-i+2), & \text{if } k+2 \le i \le 2k; \end{cases}$$

$r(v_n v_3|W_E) = (1,1,0,1)$,

$$r(v_{3i+2}v_{3i+5}|W_E) = \begin{cases} (1,0,1,1), & \text{if } i=0; \\ (i+1,i,i+1,i), & \text{if } 1 \le i \le k-1; \\ (k,k,k+1,k), & \text{if } i=k; \\ (2k-i,2k-i+1,2k-i+2, \\ 2k-i+1), & \text{if } k+1 \le i \le 2k-1; \end{cases}$$

$r(v_{n-2}v_1|W_E) = (0,1,2,1)$,

$$r(v_{3i}v_{3i+3}|W_E) = \begin{cases} (i+1,i,i-1,i), & \text{if } 1 \le i \le k; \\ (k+1,k,k,k+1), & \text{if } i=k+1; \\ (2k-i+2,2k-i+1,2k-i+2, \\ 2k-i+3), & \text{if } k+2 \le i \le 2k; \end{cases}$$

and $r(v_{n-1}v_2|W_E) = (1,0,1,2)$.
We see that there are no two edges having the same representations. This shows that $\text{edim}(C_n(1,3)) \le 4$.
On the other hand, we have to show that $\text{edim}(C_n(1,3)) \ge 4$. For this purpose, we have to show that there is no edge metric generator have cardinality 3, we suppose on contrary that $\text{edim}(C_n(1,3)) = 3$ and let $W_E = \{v_1, v_i, v_j\}$. Then the Table 9 shows all order pairs of edges (e,f) for which $r(e|W_E) = r(f|W_E)$.

Conditions on i and j	(e,f)
$2 \le i \le j \le \frac{n}{2}-1$	$(v_nv_1, v_{n-2}v_1)$
$2 \le i \le j \le \frac{n}{2}$ and $j \ne \frac{n}{2}-1$	$(v_{n-1}v_n, v_{n-3}v_n)$
$2 \le i \le \frac{n}{2}-1, \frac{n}{2}+1 \le j \le n$ and $j \ne \{\frac{n}{2}+3, \frac{n}{2}+6, \frac{n}{2}+9, \ldots, n-2, n\}$	$(v_nv_1, v_{n-2}v_1)$
$2 \le i \le \frac{n}{2}, \frac{n}{2}+1 \le j \le n$ and $i \ne \frac{n}{2}-1$, also $j \ne \{\frac{n}{2}+2, \frac{n}{2}+5, \frac{n}{2}+8, \ldots, n-3, n-1\}$	$(v_{n-1}v_n, v_{n-3}v_n)$
$i = \frac{n}{2}, \frac{n}{2}+1 \le j \le n$ and $j \ne \frac{n}{2}+2$	(v_1v_2, v_1v_4)
$i = v_{\frac{n}{2}}$ and $j = v_{\frac{n}{2}+2}$	$(v_{\frac{n}{2}+2}v_{\frac{n}{2}+3}, v_{\frac{n}{2}+2}v_{\frac{n}{2}+5})$
$i = v_{\frac{n}{2}-1}, \frac{n}{2}+1 \le j \le n$ and $j \ne \frac{n}{2}+3$	(v_2v_3, v_2v_5)
$i = v_{\frac{n}{2}-1}$ and $j = v_{\frac{n}{2}+3}$	$(v_{\frac{n}{2}}v_{\frac{n}{2}+1}, v_{\frac{n}{2}+1}v_{\frac{n}{2}+2})$

Table 9: (e,f) for which $r(e|W_E) = r(f|W_E)$

In all possibilities, we get contradiction so it concludes that there is no edge metric generator of 3 vertices. Hence $\text{edim}(C_n(1,3)) = 4$, for $n \equiv 4 \pmod 6$.
Case (viii): When $n \equiv 5 \pmod 6$ and $n \ge 11$.
Let $n = 6k+5$, $k \ge 1$, $k \in \mathbf{Z}^+$ and $W_E = \{v_1, v_2, v_{\frac{n-1}{2}-1}, v_{\frac{n-1}{2}}, v_n\} \subset V(C_n(1,3))$, we have to show that W_E is an edge metric generator of $C_n(1,3)$.

For this, we give representations of each edge of $C_n(1,3)$.
Representation for edges is

$$r(v_{3i+1}v_{3i+2}|W_E) = \begin{cases} (i,i,k-i,k-i,i+1), & \text{if } 0 \le i \le k; \\ (2k-i+2,2k-i+2,i-k, \\ i-k,2k-i+1), & \text{if } k+1 \le i \le 2k+1; \end{cases}$$

$$r(v_{3i+2}v_{3i+3}|W_E) = \begin{cases} (i+1,i,k-i,k-i,i+1), & \text{if } 0 \le i \le k-1; \\ (k+1,k,1,0,k+1), & \text{if } i = k; \\ (2k-i+1,2k-i+2,i-k+1, \\ i-k,2k-i+1), & \text{if } k+1 \le i \le 2k; \end{cases}$$

$r(v_nv_1|W_E) = (0,1,k,k+1,0)$,

$$r(v_{3i}v_{3i+1}|W_E) = \begin{cases} (i,i,k-i,k-i+1,i), & \text{if } 1 \le i \le k; \\ (k+1,k+1,1,1,k+1), & \text{if } i = k+1; \\ (2k-i+2,2k-i+2,i-k, \\ i-k,2k-i+2), & \text{if } k+2 \le i \le 2k+1; \end{cases}$$

$$r(v_{3i+1}v_{3i+4}|W_E) = \begin{cases} (i,i+1,k-i-1,k-i,i+1), & \text{if } 0 \le i \le k-1; \\ (k,k+1,0,1,k+1), & \text{if } i = k; \\ (2k-i+2,2k-i+1,i-k, \\ i-k+1,2k-i+1), & \text{if } k+1 \le i \le 2k; \end{cases}$$

$r(v_{n-1}v_2|W_E) = (1,0,k+1,k,1)$,

$$r(v_{3i+2}v_{3i+5}|W_E) = \begin{cases} (i+1,i,k-i,k-i-1,i+2), & \text{if } 0 \le i \le k-1; \\ (k+1,k,1,0,k), & \text{if } i = k; \\ (2k-i+1,2k-i+2,i-k+1, \\ i-k,2k-i), & \text{if } k+1 \le i \le 2k; \end{cases}$$

$r(v_nv_3|W_E) = (1,1,k,k+1,0)$,

$$r(v_{3i}v_{3i+3}|W_E) = \begin{cases} (i+1,i,k-i,k-i+1,i), & \text{if } 1 \le i \le k; \\ (k+1,k,1,1,k), & \text{if } i = k; \\ (2k-i+1,2k-i+2,i-k+1, \\ i-k,2k-i+2), & \text{if } k+1 \le i \le 2k; \end{cases}$$

and $r(v_{n-2}v_1|W_E) = (0,1,k,k+1,1)$.
We see that there are no two edges having the same representations. This shows that $\text{edim}(C_n(1,3)) \le 5$.
On the other hand, we have to show that $\text{edim}(C_n(1,3)) \ge 5$. For this purpose, we have to show that there is no edge metric generator have cardinality 4, we

suppose on contrary that $\mathrm{edim}(C_n(1,3)) = 4$ and let $W_E = \{v_1, v_i, v_j, v_l\}$. Then the Table 10 shows all order pairs of edges (e, f) for which $r(e|W_E) = r(f|W_E)$.

Conditions on i, j and l	(e, f)
$2 \le i \le j \le l \le \frac{n}{2}$ and $j, l \neq \frac{n}{2} - 1$	$(v_nv_1, v_{n-2}v_1)$
$2 \le i \le j \le l \le \frac{n+1}{2}$, and $j, l \neq \frac{n-1}{2}$	$(v_{n-1}v_n, v_{n-3}v_n)$
$j = v_{\frac{n-1}{2}}, l = v_{\frac{n+1}{2}}, 2 \le i \le \frac{n-3}{2}$ and $i \neq 2$	$(v_{\frac{n+3}{2}}v_{\frac{n+5}{2}}, v_{\frac{n+3}{2}}v_{\frac{n+9}{2}})$
$i = 2, j = v_{\frac{n-1}{2}}$ and $l = v_{\frac{n+1}{2}}$	$(v_{n-1}v_{n-2}, v_{n-1}v_{n-4})$
$2 \le i \le j \le \frac{n-1}{2}, \frac{n+1}{2} \le l \le n$ and $l \neq \{\frac{n+1}{2}, \frac{n+1}{2}+3, \frac{n+1}{2}+6, \frac{n+1}{2}+9, \ldots, n-2, n\}$	$(v_nv_1, v_{n-2}v_1)$
$2 \le i \le j \le \frac{n-1}{2} - 1, \frac{n+1}{2} \le l \le n$ and $j \neq \frac{n-1}{2}$, also $l \neq \{\frac{n+1}{2}+2, \frac{n+1}{2}+5, \frac{n+1}{2}+8, \frac{n+1}{2}+11, \ldots, n-3, n-1\}$	$(v_{n-1}v_n, v_{n-3}v_n)$
$j = v_{\frac{n-1}{2}}, 2 \le i \le \frac{n-3}{2}, \frac{n+1}{2} \le l \le n$ and $l \neq \frac{n+3}{2}$, also $i \neq \{2, 4, 7, 10, \ldots, \frac{n-3}{2}\}$	(v_1v_2, v_1v_4)
$j = v_{\frac{n-1}{2}}, 2 \le i \le \frac{n-3}{2}, \frac{n+1}{2} \le l \le n$ and $l \neq n-1$, also $i \neq \{2, 5, 8, \ldots, \frac{n-7}{2}, \frac{n-3}{2}\}$	$(v_{\frac{n-1}{2}-1}v_{\frac{n-1}{2}}, v_{\frac{n-1}{2}-3}v_{\frac{n-1}{2}})$
$j = v_{\frac{n-1}{2}}, 2 \le i \le \frac{n-3}{2}, \frac{n+1}{2} \le l \le n$ and $l \neq \{\frac{n+1}{2}, \frac{n+1}{2}+2, \frac{n+1}{2}+5, \frac{n+1}{2}+8, \ldots, n\}$	$(v_{\frac{n-1}{2}}v_{\frac{n+1}{2}}, v_{\frac{n-1}{2}}v_{\frac{n-1}{2}+3})$

Table 10: (e, f) for which $r(e|W_E) = r(f|W_E)$

In all possibilities, we get contradiction so it concludes that there is no edge metric generator of 4 vertices. Hence $\mathrm{edim}(C_n(1,3)) = 5$, for $n \equiv 5 \pmod 6$. $\square$

4 Conclusion

In this paper, we have studied the edge metric dimension of circulant graphs $C_n(1,2)$ and $C_n(1,3)$. It is observed that the edge metric dimension of these graphs is constant and does not depend on the number of vertices.

Open Problem: Calculate the edge metric dimension of $C_n(1,2,3)$.

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